Lecture # 7

Generalized Multiplication Rule

$$P(E_1E_2\cdots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

Ex: A deck of cards is randomly shuffled into 4 piles. What is the probability that each pile has exactly one ace?

Sol: We consider the following events:

E_1 = "ace of spades is in any of the piles"

E_2 = "ace of spades and ace of hearts are in different piles"

E_3 = "the aces of spades, hearts, diamonds are all in different piles"

E_4 = "all aces in different piles

We want $P(E_1E_2E_3E_4)$ and so

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3).$$

$P(E_1) = 1$ since the ace of spades must end up somewhere.

$P(E_2|E_1)$: Consider the pile containing the ace of spades. The probability that the ace of hearts is one of the 12 other cards is $12/51$.

So $P(E_2 | E_1) = 1 - \frac{12}{51} = \frac{39}{51}$.

$P(E_3 | E_1 E_2)$: Consider the two piles containing the aces of hearts and spades. The probability that ace of diamonds is in one of the two piles is $24/50$, so

$$P(E_3 | E_1 E_2) = 1 - \frac{24}{50} = \frac{26}{50}.$$

$P(E_4 | E_1 E_2 E_3)$: Now we have three piles each with an ace. The probability that the last ace is in one of the piles is $\frac{36}{49}$, so

$$P(E_4 | E_1 E_2 E_3) = 1 - \frac{36}{49} = 13/49$$

So $P(E_1 E_2 E_3 E_4) = 1 \cdot \frac{39}{51} \cdot \frac{26}{50} \cdot \frac{13}{49}$

$= 0.105.$

Recall: Law of Total prob: $\{F_i\}$ exhaustive/mutually excl.

$$P(E) = \sum P(E | F_i) P(F_i).$$

Ex: Suppose we have 30 marbles split into three bags as follows:

Bag 1: 2 red, 4 green, 4 blue.

Bag 2: 4 red, 5 green, 1 blue.

Bag 3: 1 red, 2 green, 7 blue.

Assuming the bags are indistinguishable, what is the probability of choosing a red marble?

Let $R =$ "choose a red marble"

If the sample space is the set of marbles, then

$$B_i = \text{"choose from the } i^{th} \text{ bag"}$$

gives an exhaustive, mutually exclusive partition of S: every marble is in at least one bag and no marble is in two bags.

So $P(R) = P(R|B_1)P(B_1) + P(R|B_2)P(B_2) + P(R|B_3)P(B_3)$.

Since the bags are indistinguishable,

$$P(B_1) = P(B_2) = P(B_3) = 1/3.$$

So $P(R) = \left(\frac{2}{10}\right)(1/3) + (4/10)(1/3) + (1/10)(1/3)$

$$= \left(\tfrac{1}{3}\right)\left(\frac{2+4+1}{10}\right) = \tfrac{7}{30}.$$

Remark: Given two events E and F, the multiplication rule and law of total probability give.

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$= \frac{P(F|E)P(E)}{P(F)}$$

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^c)P(E^c)}$$

Ex: Suppose that Chief Inspector Jacques Clouseau is 60% convinced that his main suspect is guilty. Suppose new photographic evidence shows that the perpetrator of the crime has red hair.

Question: How certain should the Chief Inspector be if the suspect also has red hair?

Solution: Let G = "the suspect is guilty"
R = "the suspect has red hair".

Then $$P(G|R) = \frac{P(GR)}{P(R)}$$
$$= \frac{P(R|G)P(G)}{P(R|G)P(G) + P(R|G^c)P(G^c)}.$$

Now, it is known that at most 2% of the population has red hair.

$P(R|G) = 1$ since we have a photo of a redheaded perpetrator.

$P(G) = 0.6$.

$P(R|G^c) = 0.02$, since if the suspect is not guilty, then they have a 2% of being a red head.

$P(G^c) = 1 - 0.6 = 0.4$.

So $$P(G|R) = \frac{1 \cdot (0.6)}{1 \cdot (0.6) + (0.02)(0.4)} = 0.986.$$

So we are 98.6% sure that the suspect is guilty!!

A note on odds:

For an event A, the odds of A are defined to be

$$\frac{P(A)}{P(A^c)} = \frac{P(A)}{1-P(A)}$$

If $\frac{P(A)}{P(A^c)} = \alpha$, then we usually say the odds of A are α to 1 in favour of A.